Original Article

Long-term interest rates and consol bond valuation

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ABSTRACT This article presents a Gaussian three-factor model of the term structure of interest rates which is Markov and time-homogeneous. The model captures the whole term structure and is particularly useful in forward simulations for applications in long-term swap and bond pricing, risk management and portfolio optimization. Kalman filter parameter estimation uses EU swap rate data and is described in detail. The yield curve model is fitted to data up to 2002 and assessed by simulation of yield curve scenarios over the next 2 years. It is then applied to the valuation of callable floating rate consol bonds as recently issued by European banks to raise Tier 1 regulatory capital over the subsequent period from 2005 to 2007.
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INTRODUCTION

The literature in the area of interest rate modelling is extensive. Traditional term structure models, such as Vasicek (1977) and Cox *et al* (1985) specify the short rate process. As short-term and long-term rates are not perfectly correlated, the data are clearly inconsistent with the use of one-factor time-homogeneous models. Chan *et al* (1992) demonstrate the empirical difficulties of one-factor continuous-time specifications within the Vasicek and Cox-Ingersoll-Ross (CIR) class of models using the generalized methods of moments.

Litterman and Scheinkman (1991) find that 96 per cent of the variability of the returns of any risk-free zero-coupon bond can be explained by three factors: the *level*, *steepness* and *curvature* of the yield curve. They also point out that the 'correct model' of the term structure may involve unobservable factors. For instance, it is widely believed that changes in the Federal Reserve policy are a major source of changes in the shape of the US yield curve. Even though the Federal Reserve policy is itself observable, it is not clear how to measure its effect on the yield curve. Litterman and Scheinkman (1991) themselves used unobservable factors in their approach by applying principal component analysis.

Most term-structure models, such as those of Ho and Lee (1986), Hull and White (1990) and Heath *et al* (1992), are specified using the risk-neutral measure corresponding to a complete market. This makes them appropriate for relative-pricing applications, but inappropriate for forward simulations, which needs to take place under the market measure in an incomplete market. An exception is Rebonato *et al* (2005) who focus on yield curve evolution under the market measure and present a semi-parametric method to explain the yield curve evolution. Ho and Lee (1986) and Heath *et al* (1992) introduced a new approach to interest rate modelling in which they fit the initial term structure exactly. Duffie and Kan (1996) developed a general theory for multifactor affine versions of these models with coefficients obtained analytically. The book by James and Webber (2000) gives a comprehensive summary of development to 2000. See also Brigo and Mercurio (2007) and Wu (2009) for more recent summaries.

In spite of significant theoretical achievements, there are still difficulties with the long-term forecasting of future yields. The affine arbitrage-free version of the Nelson-Siegel (1987) model by Christensen *et al* (2007) investigates the gap between the theoretically rigorous risk-neutral models used for pricing and the empirical tractability required by econometricians for forecasting and offers some improvements in forecasting performance.

In our research, we focus on the development of a model that allows simulation of long-term scenarios for the yield curve, which include the market prices of risk in the dynamics (Medova *et al*, 2006). Historically low interest rates in recent years have emphasized the importance of accurately valuing long-term guarantees (Wilkie *et al*, 2004; Dempster *et al*, 2006). The asset liability management of funds behind the guaranteed return products offered by pension providers must be based on yield curve modelling (Dempster *et al*, 2007, 2009). Banks also face new pricing challenges because of the increased demand for long-maturity derivatives to hedge insurance and pension liabilities (for liability-driven investment) and therefore require good long-term interest rate models.

Figure 1 shows the development over time of short- and long-term interest rates in the Eurozone for the period 1997–2002. Figure 2 plots the weekly standard deviations of the yields over the same period.

In this article, we focus on a term structure model with the following characteristics:

- The model is set in a continuous-time framework. This allows implementation in discrete time with any length of time step Δt without the need to construct a new model each time we change Δt. This is an important requirement for the flexibility of forward simulations.

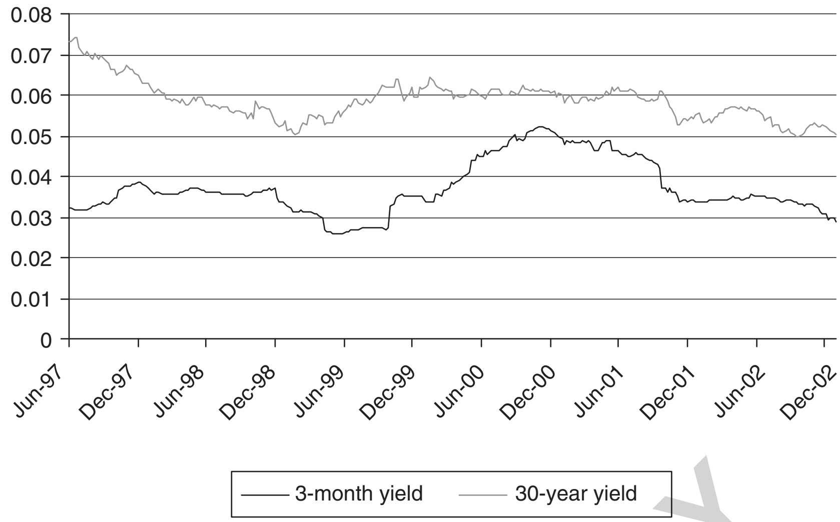

Figure 1: Three-month and thirty-year EU yields for the period June 1997 – December 2002.

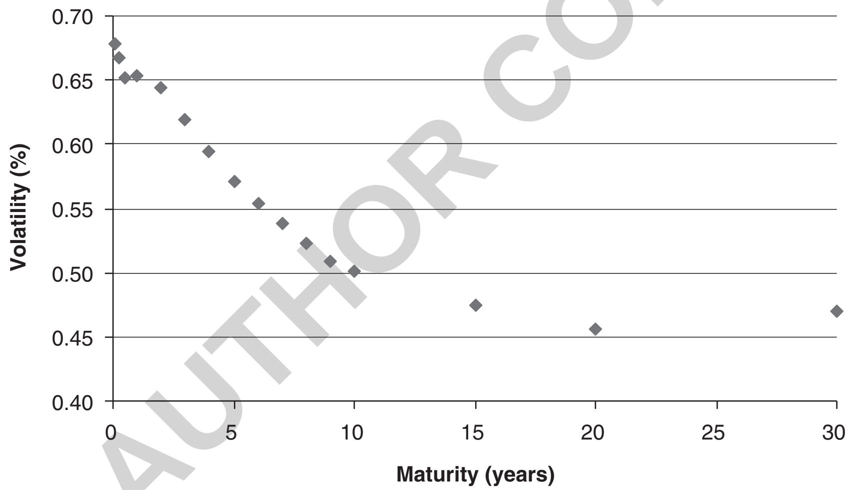

Figure 2: Weekly standard deviation of yields for the period June 1997 – Dec 2002.

- Interest-rate dynamics are consistent with what we observe in historical data.
- The affine class model has a closed-form solution for bond pricing, permitting straightforward analytical calculation of bond prices in forward simulation.
- The short rate is mean-reverting.
- The model permits a tractable method of estimation and calibration.
- The model is flexible enough to give rise to a range of different yield curve shapes and dynamics (steepening, flattening, yield curve inversion and so on).

The remainder of the article is structured as follows. In the next section, the three-factor Gaussian term structure model is introduced and a closed-form solution for bond prices is derived. The section after that discusses the state-space formulation of the model and the estimation of its parameters using the Kalman filter and numerical likelihood maximization. The data and empirical analyses, focussing on fitting the data as well as on the simulation potential of the model, are presented in the subsequent section. The penultimate section applies the three-factor model to the pricing of a representative consol bond and the final section concludes the article.

THREE-FACTOR TERM STRUCTURE MODEL

The term structure model presented in this article is driven by three factors and can be viewed as an extension to the generalized Vasicek model of Langetieg (1980), which includes a third factor. The first two factors[1] **X** and **Y** satisfy the standard Vasicek stochastic differential equations with *mean reversion rates* λ_X and λ_Y and *levels* μ_X/λ_X and μ_Y/λ_Y, respectively. The innovation of our model is in the treatment of the instantaneous short rate **R**. The mean reversion of **R** with rate k has a level that is *stochastic* rather than deterministic and depends on the level of the other two factors **X** and **Y** driving the model. The **X** and **Y** factors may be interpreted, respectively, as a long rate and (minus) the slope of the yield curve from a perceived (instantaneous) short rate $\mathbf{R}^* := \mathbf{X} + \mathbf{Y}$.

Risk-neutral measure

Starting from the formulation of the model under the risk-neutral measure Q, we have the following three stochastic differential equations (SDEs) for the factors

$$\mathrm{d}\mathbf{X}_t = (\mu_X - \lambda_X X_t)\mathrm{d}t + \sigma_X \mathrm{d}\widetilde{\mathbf{W}}_t^X, \quad (1)$$

$$\mathrm{d}\mathbf{Y}_t = (\mu_Y - \lambda_Y Y_t)\mathrm{d}t + \sigma_Y \mathrm{d}\widetilde{\mathbf{W}}_t^Y, \quad (2)$$

$$\mathrm{d}\mathbf{R}_t = k(X_t + Y_t - R_t)\mathrm{d}t + \sigma_R \mathrm{d}\widetilde{\mathbf{W}}_t^R, \quad (3)$$

where the $\mathrm{d}\widetilde{\mathbf{W}}$ terms are correlated. Factoring the covariance matrix of the $\mathrm{d}\widetilde{\mathbf{W}}$ terms using a Cholesky decomposition into the product of a transposed upper and a lower triangular square root matrix results in a new formulation of the form

$$\mathrm{d}\mathbf{X}_t = (\mu_X - \lambda_X X_t)\mathrm{d}t + \sum_{i=1}^{3} \sigma_{X_i} \mathrm{d}\mathbf{Z}_t^i, \quad (4)$$

$$\mathrm{d}\mathbf{Y}_t = (\mu_Y - \lambda_Y Y_t)\mathrm{d}t + \sum_{i=1}^{3} \sigma_{Y_i} \mathrm{d}\mathbf{Z}_t^i, \quad (5)$$

$$\mathrm{d}\mathbf{R}_t = k(X_t + Y_t - R_t)\mathrm{d}t + \sum_{i=1}^{3} \sigma_{R_i} \mathrm{d}\mathbf{Z}_t^i, \quad (6)$$

where the $\mathrm{d}\mathbf{Z}$ terms are uncorrelated.

Closed-form solution

The solution follows the usual steps. We first solve the SDEs for **X, Y** and **R** to obtain the price of a zero-coupon bond at time t paying 1 at time T

$$P(t,T) = \mathbb{E}_t^Q \left\{ \exp\left(-\int_t^T \mathbf{R}_s \mathrm{d}s \right) \right\}, \quad (7)$$

where $\mathbb{E}_t^Q$ denotes the expectation under the risk-neutral measure Q conditional on the information at time t. As $\mathbf{R}_s$ is normally distributed in our model, we can use the moment-generating function for the normal distribution to rewrite (7) as

$$P(t,T) = \exp\left\{ \mathbb{E}_t^Q \left(-\int_t^T \mathbf{R}_s \mathrm{d}s \right) + \frac{1}{2}\mathrm{var}_t^Q \left(-\int_t^T \mathbf{R}_s \mathrm{d}s \right) \right\}, \quad (8)$$

where var_t^Q denotes the conditional variance under Q. Integrating the solution of the SDE for **R** and taking the expectation and variance of the result gives expressions for the two terms in (8) involving (to simplify notation) the parameters

$$\begin{aligned} m_{X_i} &:= -\frac{k\sigma_{X_i}}{\lambda_X(k-\lambda_X)}, \\ m_{Y_i} &:= -\frac{k\sigma_{Y_i}}{\lambda_Y(k-\lambda_Y)}, \\ n_i &:= \frac{\sigma_{X_i}}{k-\lambda_X} + \frac{\sigma_{Y_i}}{k-\lambda_Y} - \frac{\sigma_{R_i}}{k}, \\ p_i &:= -(m_{X_i} + m_{Y_i} + n_i). \end{aligned} \tag{9}$$

Hence (see for example, Medova *et al*, 2006),

$$\begin{aligned} P(t,T) &= e^{-y_{t,T}(T-t)} \\ &= \exp\{-A(t,T)R_t - B(t,T)X_t \\ &\quad -C(t,T)Y_t - D(t,T)\} \end{aligned} \tag{10}$$

with corresponding yield to maturity

$$\mathbf{y}_{t,T} = \frac{A(t,T)R_t + B(t,T)X_t + C(t,T)Y_t + D(t,T)}{T-t}, \tag{11}$$

where

$$A(t,T) = \frac{1}{k}(1 - e^{-k(T-t)}), \tag{12}$$

$$\begin{aligned} B(t,T) = \frac{k}{k-\lambda_X}\bigg\{&\frac{1}{\lambda_X}(1 - e^{-\lambda_X(T-t)}) \\ &-\frac{1}{k}(1 - e^{-k(T-t)})\bigg\}, \end{aligned} \tag{13}$$

$$\begin{aligned} C(t,T) = \frac{k}{k-\lambda_Y}\bigg\{&\frac{1}{\lambda_Y}(1 - e^{-\lambda_Y(T-t)}) \\ &-\frac{1}{k}(1 - e^{-k(T-t)})\bigg\}, \end{aligned} \tag{14}$$

$$\begin{aligned} D(t,T) = &\left(T - t - \frac{1}{k}(1 - e^{-kT})\right)\left(\frac{\mu_X}{\lambda_X} + \frac{\mu_Y}{\lambda_Y}\right) \\ &- \frac{\mu_X}{\lambda_X}B(t,T) - \frac{\mu_Y}{\lambda_Y}C(t,T) \\ &- \frac{1}{2}\sum_{i=1}^{3}\bigg\{\frac{m_{X_i}^2}{2\lambda_X}(1 - e^{-2\lambda_X(T-t)}) \\ &+ \frac{m_{Y_i}^2}{2\lambda_Y}(1 - e^{-2\lambda_Y(T-t)}) \\ &+ \frac{n_i^2}{2k}(1 - e^{-2k(T-t)}) + p_i^2(T-t) \\ &+ \frac{2m_{X_i}m_{Y_i}}{\lambda_X+\lambda_Y}(1 - e^{-(\lambda_X+\lambda_Y)(T-t)}) \\ &+ \frac{2m_{X_i}n_i}{\lambda_X+k}(1 - e^{-(\lambda_X+k)(T-t)}) \\ &+ \frac{2m_{X_i}p_i}{\lambda_X}(1 - e^{-\lambda_X(T-t)}) \\ &+ \frac{2m_{Y_i}n_i}{\lambda_Y+k}(1 - e^{-(\lambda_Y+k)(T-t)}) \\ &+ \frac{2m_{Y_i}p_i}{\lambda_Y}(1 - e^{-\lambda_Y(T-t)}) \\ &+ \frac{2n_ip_i}{k}(1 - e^{-k(T-t)})\bigg\}. \end{aligned} \tag{15}$$

Market measure

Bond pricing is achieved under the *risk-neutral* measure Q. However, for the model to be used for forward simulations, we need to adjust the set of SDEs so that we capture the model dynamics under the *market* (or real-world) measure P by adding a *risk premium* to each drift term. The risk premium is given by the *market price of risk* γ times the quantity of risk, and it is generally assumed in a Gaussian specification that the quantity of risk is given by the *volatility* of each factor. We assume that the market prices of risk are independent of the time to maturity of the bond and are not functionally dependent on the factor being modelled.

The set of processes under the market measure thus satisfy

$$\begin{aligned} d\mathbf{X}_t = &(\mu_X - \lambda_X X_t + \gamma_X\sigma_X)dt \\ &+ \sigma_X d\widetilde{\mathbf{W}}_t^X, \end{aligned} \tag{16}$$

$$d\mathbf{Y}_t = (\mu_Y - \lambda_Y Y_t + \gamma_Y \sigma_Y)dt + \sigma_Y d\widetilde{\mathbf{W}}_t^Y, \quad (17)$$

$$d\mathbf{R}_t = \{k(X_t + Y_t - R_t) + \gamma_R \sigma_R\}dt + \sigma_R d\widetilde{\mathbf{W}}_t^R, \quad (18)$$

where all three factors contain a market price of risk γ in volatility units.

ESTIMATION PROCEDURE

The estimation of affine term structure models is known to be problematic because of the existence of numerous model likelihood maxima with essentially identical fit to the data (Kim and Orphanides, 2005). Babbs and Nowman (1999) applied the Kalman filter to estimate the two-factor generalized Vasicek model. Some other examples of the literature on filtering methods are Chen and Scott (1993), De Jong (2000), De Jong and Santa-Clara (1999), Geyer and Pichler (1999) and Duffee (2002). Most of these papers analyse multi-factor versions of the CIR model using mutually independent factors. De Jong (2000) extends this approach to the more general class of affine models proposed by Duffie and Kan (1996).

Kalman filter

Here we describe in detail the Kalman filter estimation procedure (Harvey, 1989), for our three-factor yield curve model in state-space form, which simultaneously integrates time series and cross-sectional aspects of the model. It also allows the identification of the market prices of interest rate risk critical for forward simulation.

The general state-space form applies to multivariate time series. The N observable variables $\mathbf{y}_t$ at time t (here, zero-coupon bond yields of various maturities) are related to a vector $\boldsymbol{\alpha}_t$ known as the *state vector* (here our three yield curve factors) via a *measurement equation*

$$\mathbf{y}_t = \mathrm{Z}\alpha_t + \mathrm{d} + \varepsilon_t, \quad t = 1, \ldots, T, \quad (19)$$

where Z is an $N \times m$ matrix, α_t is an $m \times 1$ vector, d and ε_t are $N \times 1$ vectors and the error term is assumed to consist of serially uncorrelated disturbances with mean zero and covariance matrix H, that is,

$$\mathbb{E}(\boldsymbol{\varepsilon}_t) = 0 \qquad \mathrm{var}(\boldsymbol{\varepsilon}_t) = \mathrm{H}. \quad (20)$$

In general, Z, d and H may depend on t.

Even though the elements of the state $\boldsymbol{\alpha}_t$ are unobservable, they are known to follow a first-order Markov process specified by the *transition equation*

$$\boldsymbol{\alpha}_t = \mathrm{A}\alpha_{t-1} + \mathrm{c} + \mathrm{S}\boldsymbol{\eta}_t, \quad t = 1, ., T, \quad (21)$$

where A is an $m \times m$ matrix, c an $m \times 1$ vector, S an $m \times g$ matrix and $\boldsymbol{\eta}_t$ a $g \times 1$ vector of serially uncorrelated disturbances with mean zero and covariance matrix Q, that is,

$$\mathbb{E}(\boldsymbol{\eta}_t) = 0, \qquad \mathrm{var}(\boldsymbol{\eta}_t) = \mathrm{Q}. \quad (22)$$

Again, in general, A, c and S may depend on t, however, we will treat here the *time-homogeneous* case appropriate to the long term.[2]

Two further assumptions will be required to complete the state-space formulation:

- The initial state vector $\boldsymbol{\alpha}_0$ has mean a_0 and covariance matrix P_0, that is

$$\mathbb{E}(\boldsymbol{\alpha}_0) = a_0, \qquad \mathrm{var}(\boldsymbol{\alpha}_0) = \mathrm{P}_0. \quad (23)$$

- The disturbance terms $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\eta}_t$ are uncorrelated with each other in all time periods and uncorrelated with the initial state, that is,

$$\mathbb{E}(\boldsymbol{\varepsilon}_t\boldsymbol{\eta}_s') = 0, \quad \text{for all} \quad s, t = 1, \ldots, T \quad (24)$$

and

$$\mathbb{E}(\boldsymbol{\varepsilon}_t\boldsymbol{\alpha}_0') = 0 \quad \mathbb{E}(\boldsymbol{\eta}_t\boldsymbol{\alpha}_0') = 0 \quad t = 1, \ldots, T. \quad (25)$$

The important concept behind the state-space formulation is this separation of the noise driving the system dynamics $\boldsymbol{\eta}_t$ and the observational noise $\boldsymbol{\varepsilon}_t$.

The *Kalman filter* is applied recursively in order to compute the optimal estimator of the state vector at time t given all the information currently available, which consists of the observations up to and including y_t. Assuming a Gaussian state space, the disturbances and the initial state vector will be normally distributed.

In a state-space model, the system matrices depend on a set of unknown parameters (in our case 14) referred to as *hyper-parameters* and defined in Table 1. Using the Kalman filter to construct the likelihood function and then maximizing it using a suitable numerical optimization procedure, we can carry out maximum likelihood estimation of the hyper-parameters. The joint probability of a set of T observations can be expressed in terms of conditional distributions. For a multivariate normal distribution we have

$$L(y;\varphi)=\prod_{t=1}^{T}p(y_t|Y_{t-1}), \qquad (26)$$

where $p(y_t|Y_{t-1})$ is the distribution of $\mathbf{y}_t$ conditional on the information at time $t-1$, that is $Y_{t-1}=(y_{t-1},y_{t-2},\ldots,y_1)'$. As we have a Gaussian model, we can write the log-likelihood function in *prediction error decomposition form* as

$$\log L(\varphi)=-\frac{NT}{2}\log 2\pi -\frac{1}{2}\sum_{t=1}^{T}\log|F_t|-\frac{1}{2}\sum_{t=1}^{T}v_t' F_t^{-1} v_t, \qquad (27)$$

where F_t is estimated by the covariance matrix obtained from the Kalman filter as

$$F_t = ZP_{t|t-1}Z' + H \qquad (28)$$

and v_t is the vector of *prediction errors* given by

$$\begin{aligned} v_t &= y_t - \tilde{y}_{t|t-1} \\ &= Z(\alpha_t - \alpha_{t|t-1}) + \varepsilon_t. \end{aligned} \qquad (29)$$

Together with the following two equations, (28) and (29) form the *measurement update equations*

$$a_t = a_{t|t-1} + P_{t|t-1}Z'F_t^{-1}v_t, \qquad (30)$$

$$P_t = P_{t|t-1} - P_{t|t-1}Z'F_t^{-1}P_{t|t-1}. \qquad (31)$$

Therefore, first we specify starting values for the parameters. With these starting values, we run the Kalman filter to obtain estimated yields and a time series for the unobserved state variables. Next, the parameters are estimated by maximizing the log-likelihood using the state variable path estimates as observations. The optimized parameter values are then used as the starting values for the next iteration of the Kalman

Table 1: Estimated parameters using the Kalman filter

Euro data		*Estimated value*	*SE*
Long-term risk-neutral mean *X*	μ_X/λ_X	0.199	1.69E-04
Long-term risk-neutral mean *Y*	μ_Y/λ_Y	−0.134	1.69E-04
Speed of mean reversion *X*	λ_X	0.161	1.03E-03
Speed of mean reversion *Y*	λ_Y	1.332	6.87E-03
Speed of mean reversion *R*	k	0.117	1.64E-03
Volatility *X*	σ_X	0.030	1.89E-04
Volatility *Y*	σ_Y	0.186	9.80E-04
Volatility *R*	σ_R	0.006	2.26E-04
Correlation *X* and *Y*	ρ_{XY}	−0.642	6.94E-03
Correlation *X* and *R*	ρ_{XR}	0.177	1.82E-02
Correlation *Y* and *R*	ρ_{YR}	−0.540	1.81E-02
Market price of risk for *X*	γ_X	0.556	3.91E-03
Market price of risk for *Y*	γ_Y	−1.017	5.50E-03
Market price of risk for *R*	γ_R	0.096	1.65E-02

filter. This loop continues until we obtain the optimal parameter estimates by this generalized EM algorithm (Dempster *et al*, 1977). The calibration code is implemented in C++ and the optimization is performed using a combination of global (Direct, see Jones *et al*, 1993) and local (approximate) conjugate direction (Powell, 1964) or derivative-free quasi-Newton (NAG BFGS, used in the section 'Pricing consol bonds') numerical algorithms.

The starting values for the Kalman filter are given by the mean and the covariance of the unconditional distribution of the stationary state vector. The state vector is stationary if c and A are time invariant and $|\lambda(\mathrm{A})|<1$, where $\lambda(\mathrm{A})$ is the leading eigenvalue of A. In this case, the mean a_0 is given by the unique solution to

$$\begin{aligned} a_0 &= \mathrm{A}a_0 + \mathrm{c} \quad \text{given by} \\ a_0 &= (\mathrm{I}-\mathrm{A})^{-1}\mathrm{c} \end{aligned} \tag{32}$$

and the covariance matrix P_0 will be given by the unique solution to the *Riccati equation*

$$\begin{aligned} \mathrm{P}_0 =& \mathrm{AP}_0\mathrm{A}' + \mathrm{SQS}' \quad \text{given by} \quad \mathrm{vec}(\mathrm{P}_0) \\ =& (\mathrm{I}-\mathrm{A}\otimes\mathrm{A})^{-1}\mathrm{vec}(\mathrm{SQS}'). \end{aligned} \tag{33}$$

State-space form

In our case, the observable variables are given by risk-free (Treasury) yields of different maturities, and are related to the vector of unobservable state variables (X, Y, R) via the measurement equation. The measurement equation is obtained using (11) and adding serially and cross-sectionally uncorrelated disturbances with mean zero to take into account non-simultaneity of the observations, errors in the data and so on. The unobservable state variables are generated via the transition equations, which in our case are given by the discretized versions of (1), (2) and (3), using Euler's first-order approximation,[3] that is

$$\begin{aligned} \mathbf{X}_{t+\Delta t} =& X_t + (\mu_X - \lambda_X X_t \\ &+ \gamma_X\sigma_X)\Delta t + \sigma_X\sqrt{\Delta t}\,\boldsymbol{\eta}_{t,X}, \end{aligned} \tag{34}$$

$$\begin{aligned} \mathbf{Y}_{t+\Delta t} =& Y_t + (\mu_Y - \lambda_Y Y_t + \gamma_Y\sigma_Y)\Delta t \\ &+ \sigma_Y\sqrt{\Delta t}\,\boldsymbol{\eta}_{t,Y}, \end{aligned} \tag{35}$$

$$\begin{aligned} \mathbf{R}_{t+\Delta t} =& R_t + (k(X_t + Y_t - R_t) \\ &+ \gamma_R\sigma_R)\Delta t + \sigma_R\sqrt{\Delta t}\,\boldsymbol{\eta}_{t,R}. \end{aligned} \tag{36}$$

In matrix form, the transition equations can be written as

$$\begin{pmatrix} \mathbf{X}_t \\ \mathbf{Y}_t \\ \mathbf{R}_t \end{pmatrix} = \mathrm{A}\begin{pmatrix} X_{t-\Delta t} \\ Y_{t-\Delta t} \\ R_{t-\Delta t} \end{pmatrix} + \mathrm{c} + \mathrm{S}\boldsymbol{\eta}_t, \tag{37}$$

where

$$\mathrm{A} := \begin{pmatrix} 1-\lambda_X\Delta t & 0 & 0 \\ 0 & 1-\lambda_Y\Delta t & 0 \\ k\Delta t & k\Delta t & 1-k\Delta t \end{pmatrix}, \tag{38}$$

$$\mathrm{c} := \begin{pmatrix} (\mu_X + \gamma_X\sigma_X)\Delta t \\ (\mu_Y + \gamma_Y\sigma_Y)\Delta t \\ \gamma_R\sigma_R\Delta t \end{pmatrix}, \tag{39}$$

$$\mathrm{S} := \begin{pmatrix} \sigma_X\sqrt{\Delta t} & 0 & 0 \\ 0 & \sigma_Y\sqrt{\Delta t} & 0 \\ 0 & 0 & \sigma_R\sqrt{\Delta t} \end{pmatrix} \tag{40}$$

and $\boldsymbol{\eta}_t$ is a vector with serially uncorrelated disturbances satisfying

$$\begin{aligned} \mathbb{E}(\boldsymbol{\eta}_t) &= 0 \\ \mathrm{var}(\boldsymbol{\eta}_t) &= \begin{pmatrix} 1 & \rho_{XY} & \rho_{XR} \\ \rho_{XY} & 1 & \rho_{YR} \\ \rho_{XR} & \rho_{YR} & 1 \end{pmatrix}. \end{aligned} \tag{41}$$

In the current literature, several approaches have been adopted to estimate the covariance matrix of the measurement errors. For example, De Jong and Santa Clara (1999) used a spherical covariance matrix, $\mathrm{H} = h\mathrm{I}$, whereas Babbs and Nowman (1999) use a diagonal matrix. De Jong (2000) uses a full covariance matrix. We adopt a diagonal covariance matrix approach, optimizing likelihood using one-at-a-time search with the parameters divided into two groups: in the first search, the model parameters are optimized followed by the minimization of the measurement errors in the second search. This process is repeated until convergence. The one-at-a-time search method is preferred over the full optimization with 14 model parameters and 16 measurement errors because of the scale of the optimization problem in the combined case. Even though the full covariance matrix is to be highly preferred, we have avoided this specification, because using yields of 16 different maturities would result in 136 noise parameters to be estimated.

Estimation results

For our empirical analysis, yields on ordinary (par) fixed-for-floating rate Euro swap contracts[4] are used as data. As the swap market is highly liquid with many par swaps traded every day, it is possible to obtain rates for a set of swaps with *constant* maturities from 1 to 30 years from the market. From the market swap rates, a swap curve that gives the rates for constant maturity swaps (CMS) of *all* durations may be constructed each day. Dai and Singleton (2000) point out that these yields are preferable for analysis for the following reasons. The swap markets provide 'constant maturity' yield data, whereas in the Treasury market the maturities of 'constant maturity' yields are only approximately constant or the data represent interpolated series. In addition, the on-the-run (that is, just purchased at auction) treasuries that are often used in empirical studies are typically on 'special' (haircut) in the repo market to which they are immediately (although temporarily) sold. Therefore, strictly speaking, the Treasury data should be adjusted for repo specials before any empirical analysis. Unfortunately, the requisite data for making these adjustments are not readily available, and consequently such adjustments are rarely made.

For estimation and calibration purposes, we first use weekly 1-, 3- and 6-month EU LIBOR and Euro swap data for the period from June 1997 to December 2002 (a total of 292 time points) for 16 different yields with maturities equal to 1, 3 and 6 months and 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20 and 30 years. The sample period was determined on one end by the unavailability of reliable long-term swap data for years before 1997 and on the other end by its use for backtests over the difficult 2003–2004 period in the bond markets.[5] We interpolate the swap curve linearly to obtain swap rates at all maturities and then use the data recursively from the 1-month rate to back out a zero-coupon bond yield curve from the basic swap pricing equation for each week.[6] This derived data is the input for model calibration. The estimation results are presented in Table 1 all have plausible values. Bearing in mind that the factor **y** is a *negative* yield curve slope, between a market expected short rate and the long rate, all the signs in Table 1 are as expected. All parameter estimates are statistically significant at the 1 per cent level, unlike the estimates found by Babbs and Nowman (1999), who looked at Kalman filtering generalized Vasicek models. However, they only used yields of eight different maturities and Geyer and Pichler (1999) show that a larger number of maturities is important to improve the precision of the parameter estimates. Shocks to the long rate and the expected yield curve slope decay here with half lives over 4 years and about 6 months, respectively.

Table 2 provides the estimated standard deviations $\sqrt{h_i}$ of the measurement errors, where h_i is the ith diagonal element of the covariance matrix H. In particular, these standard deviations range from less than 1 basis point for the 7-year yield to 24 basis points for the 30-year yield. These measurement errors are all significant at the

1 per cent level and compare in magnitude to those in Babbs and Nowman (1999) and very favourably to studies by, for example Chen and Scott (1993) and Geyer and Pichler (1999), who both estimate the multifactor CIR model on US data.

A general limitation of affine yield curve models with mean reversion is that the volatility of long rates tends to decay too rapidly with maturity relative to that exhibited by market data. Our use of maturity-specific volatilities for the measurement errors compensates for this effect. Indeed, the standard deviation of the measurement errors for the 15, 20 and 30 years' rates shown in Table 2 are significantly greater than those for the other maturities. More generally, and similar to Geyer and Pichler (1999), the error standard deviations exhibit a distinct U-shaped pattern as depicted in Figure 3. A possible explanation for this might be that the observed data for medium range maturities are highly correlated and therefore easier to fit. The short rate behaviour in Figure 3 also indicates that the use of the 1-month yield as a proxy for the instantaneous short rate is likely to give rise to problems. In general, 1-month and 6-month LIBOR rate measurement errors appear inconsistent with those for rates derived from the swap data for maturities in years, probably because of liquidity factors.

Table 2: Measurement errors

	Maturity	*Estimated value*	*SE*
$3\sqrt{h_1}$	1 month	1.57E-03	6.63E-05
$3\sqrt{h_2}$	3 months	8.64E-04	3.81E-05
$3\sqrt{h_3}$	6 months	1.55E-04	3.19E-05
$3\sqrt{h_4}$	1 year	6.71E-04	2.96E-05
$3\sqrt{h_5}$	2 years	5.08E-04	2.15E-05
$3\sqrt{h_6}$	3 years	2.85E-04	1.21E-05
$3\sqrt{h_7}$	4 years	1.49E-04	7.03E-06
$3\sqrt{h_8}$	5 years	4.96E-05	4.59E-06
$3\sqrt{h_9}$	6 years	6.58E-05	2.89E-06
$3\sqrt{h_{10}}$	7 years	1.00E-05	3.83E-06
$3\sqrt{h_{11}}$	8 years	9.44E-05	4.10E-06
$3\sqrt{h_{12}}$	9 years	1.75E-04	7.63E-06
$3\sqrt{h_{13}}$	10 years	2.94E-04	1.28E-05
$3\sqrt{h_{14}}$	15 years	7.45E-04	3.14E-05
$3\sqrt{h_{15}}$	20 years	1.23E-03	5.32E-05
$3\sqrt{h_{16}}$	30 years	2.37E-03	1.03E-04

Factor loadings

Similar to Babbs and Nowman (1999), we also look at the factor loadings of our three-factor model as a function of maturity to determine the nature of the factors calculated by the Kalman filter. The resulting curve for each factor represents the change in yield caused by a shock to that factor of one standard deviation magnitude so that all shocks are equally likely events (Litterman and Scheinkman, 1991). As factor loadings correspond to orthogonal Brownian

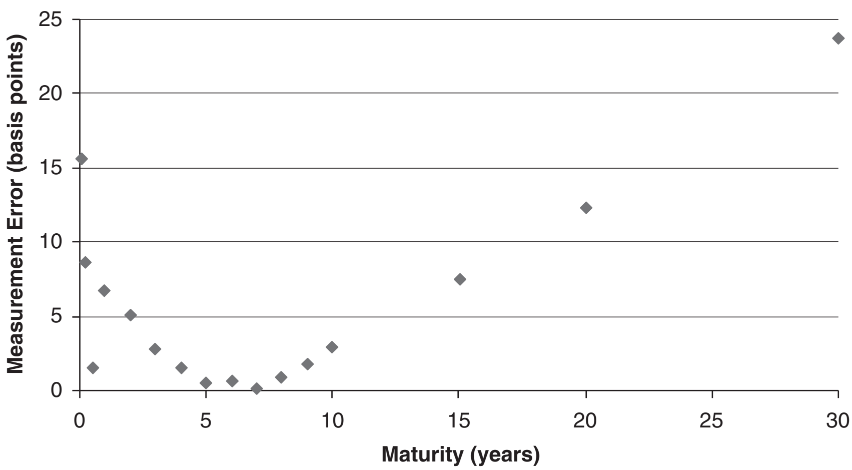

Figure 3: Measurement error of the fitted yields.

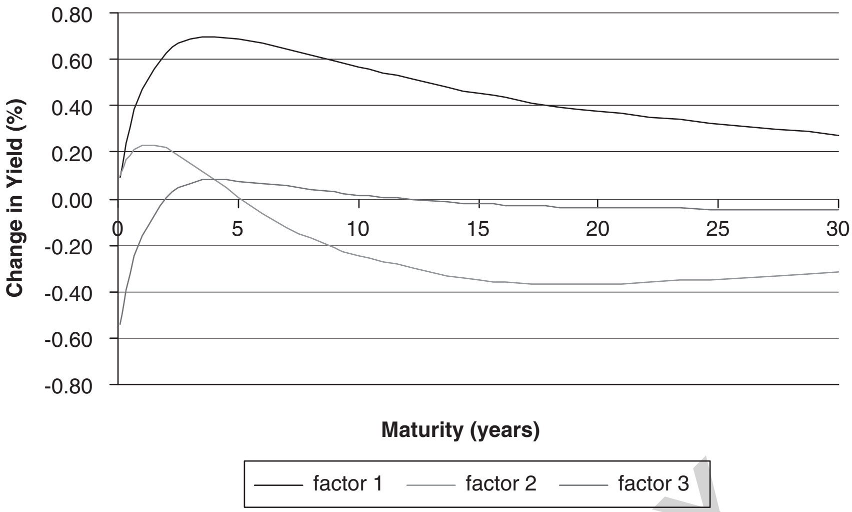

Figure 4: Factor loadings of the three-factor model.

motions, rather than those with correlated innovations, we first use Cholesky decomposition, as described in the section 'Three-factor term structure model', to transform the SDEs. For comparison with Babbs and Nowman (1999), we also impose the following three additional restrictions: the second factor has zero impact on the term structure at approximately the 5-year maturity and the third factor loading disappears at about 2 and 12 years. This gives a set of nine equations in the nine volatility parameters of (4), (5) and (6).

Figure 4 plots the factor loadings for the three-factor model. Although Babbs and Nowman found that their third factor loading had a negligible effect, we find all three factors have a significant impact on the yields of all maturities. We also find that the range of the impact of the factors **R**, **Y** and **X** on the yields is similar to that found by Litterman and Scheinkman (1991), using principal component analysis on weekly US data with these three factors interpreted as *level*, *curvature* and *steepness,* respectively.

SIMULATION

One of the objectives of this article is to propose a term structure model that is tractable in forward simulations of closed form yields through its factors but can still capture the salient features of the yield curve.

Yield curve statistics

To evaluate our model initially, we performed an out-of-sample backtest over 2003. Using the historical 52 weekly data points for the yields in 2003, we calculated the mean level and the weekly standard deviation for each of the 16 maturities. We then simulated forward from January 2003 to beginning January 2004 using the parameter estimates given in Table 1. In total, 500 scenarios were generated and for each scenario the mean and standard deviation, over time for the 16 maturities, was calculated. Averaging over all scenarios finally gives an average mean and standard deviation for the simulated yields.

Figure 5 plots the mean levels of the yields for both the historical and the simulated data and Figure 6 similarly plots the standard deviations. As can be observed from Figure 5, the two sets of means closely match each other. Figure 6 shows that the simulated standard deviations slightly overestimate the historical ones. However, yields were considerably less volatile in 2003 than over

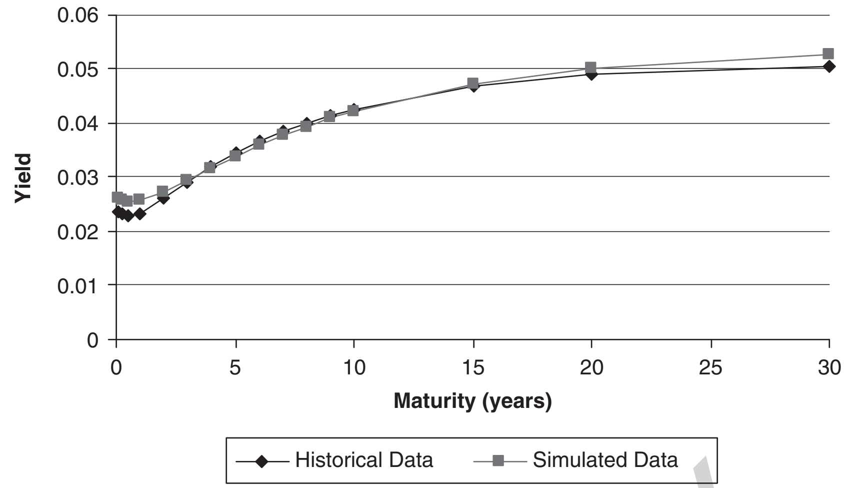

Figure 5: Mean level of yields over 2003 for historical and simulated data.

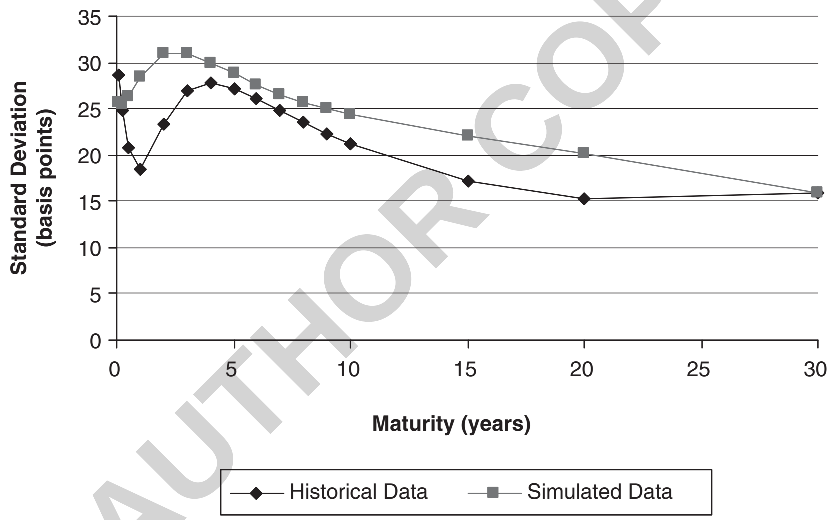

Figure 6: Weekly standard deviation of yields over 2003 for historical and simulated data.

the 1997–2002 in-sample period (cf. Figure 2), which would explain this discrepancy.

Yield curve dynamics

Another objective of this work was to develop a model that is able to simulate the various yield curve dynamics encountered in practice, for example steepening, flattening and inversion. Figures 7 and 8 show historical yields up to 2002, followed by simulated yields for 2 years to end 2004 on specific yield curve scenarios selected from the 500 simulated scenarios.[7] Figure 7 demonstrates that the model can simulate yield curve steepening and flattening, whereas Figure 8 demonstrates that it can simulate yield curve inversion. Indeed, both visual and statistical analysis of the 500 simulated scenarios (not presented here in the interests of brevity) demonstrate that the model's simulated dynamic behaviour is consistent with the historical behaviour of the yield curve over the out-of-sample period. This has been true for the many different applications with different time steps for which we have used it.

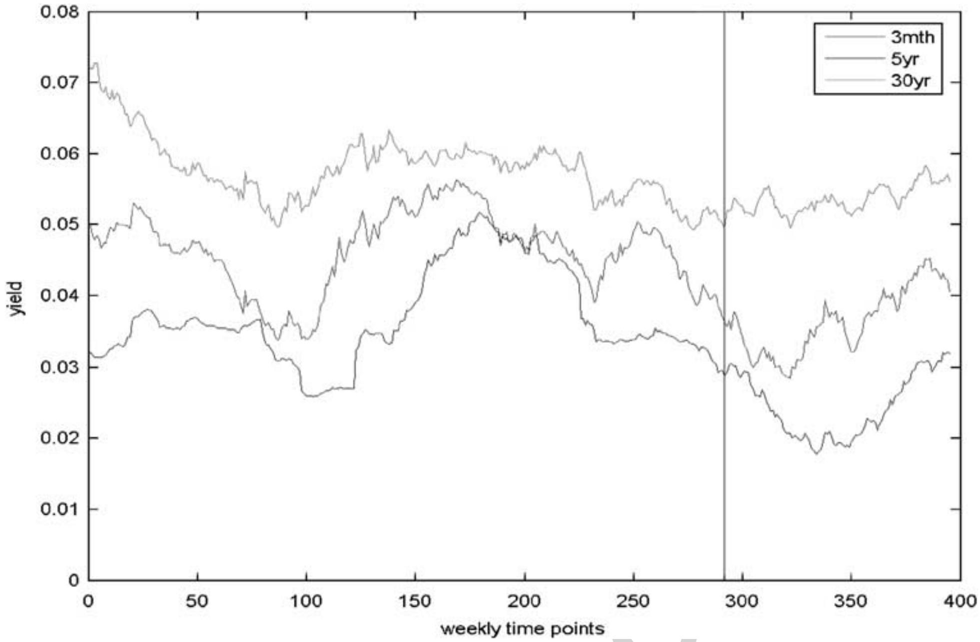

Figure 7: Forward simulation showing yield curve steepening and flattening.

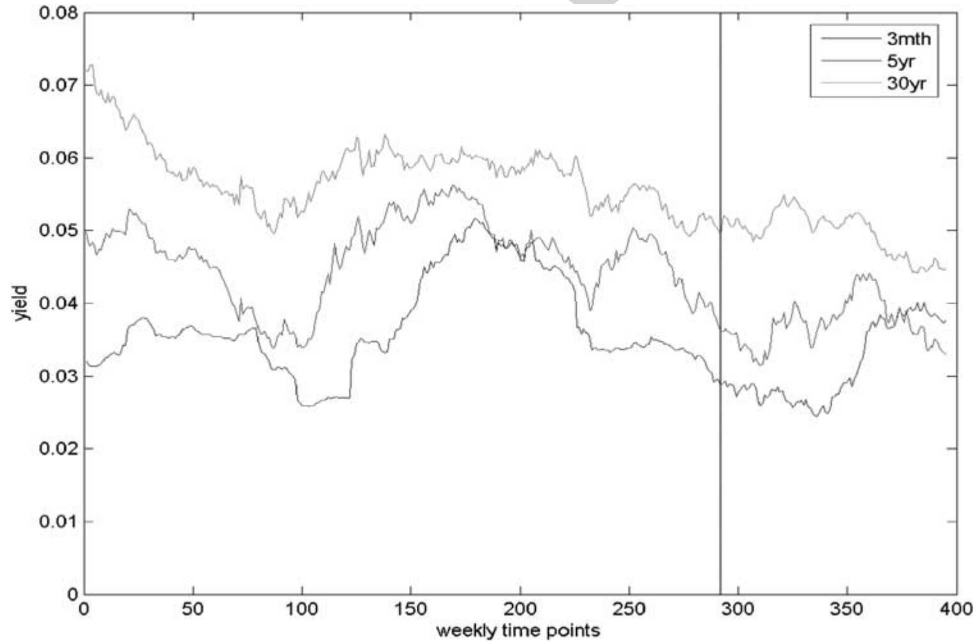

Figure 8: Forward simulation showing yield curve inversion.

Application of the yield curve model to the risk management of portfolios of guaranteed investment products involving simulations with a monthly time step may be found in Dempster *et al* (2006, 2007, 2009).

PRICING CONSOL BONDS

In 2005, a number of European banks[8] issued floating rate callable *consol* bonds as a means of raising Tier 1 regulatory capital. In the absence of the exercise of the call option by the bank at any time after a specified number of fixed interest payments, these bonds are *perpetual*, that is, they have an infinite maturity, and their holders have purchased an indefinite income stream in exchange for their capital. After the period specified, the fixed rate payable by the banks to the holder on the nominal face value of the bond is converted to a floating rate, which is a multiple of the CMS-spread, usually the difference between the 10 and 2 year CMS rates, on the interest payment fixing date. In addition, these bonds' payments normally have a cap and a floor, possibly in terms of another floating money market rate, such as 3-month EURIBOR. In the *worst* case, when the swap curve is flat (see Figure 10) and the spread negligible, the bond holder will receive the floor rate.

The details of *over-the-counter* contracts are important and we present here a representative example of a callable CMS-spread consol hybrid product used to generate Tier 1 capital by the issuing bank.

Example

Nominal (face) value	€1.5 million
Commencement date of contract	28 January 2005
Maturity	perpetual (unless called at 5 years or after)

The bank pays a fixed rate of 6 per cent per annum in arrears for 5 years. The interval between payments on 28 January each year is 1 year. At these dates, the annual floating rate payments in arrears are calculated as

$$4(CMS10_i - CMS2_i) \quad i = 6, 7, \ldots,$$

where CMS10 is the 10 year swap rate (base rate 10) and CMS2 is the 2 year swap rate (base rate 2) and ($CMS10_i - CMS2_i$) is referred to as the *spread*. This floating rate coupon is capped at 10 per cent per annum and floored at 3.5 per cent per annum. As the spread decreases to 0, the bondholder receives only the floor rate of 3.5 per cent per annum.

The expected value of this bond for the purchaser (that is, investor) is embedded in their belief about future movements of the swap curve over time. Historical swap curves illustrate that the expected and realized market rates may differ significantly. Figure 9 shows the movements of swap curves in 2005 and 2006.

It is obvious that interest rate and, more generally, swap curve forecasting rests at the centre of CMS-spread instrument pricing for both seller and purchaser. Analysis of the risk of the proposed bond depends upon the counterparties' abilities to model and simulate forward the yield and swap curves at the payment dates under the market measure. The broad movements of the spreads

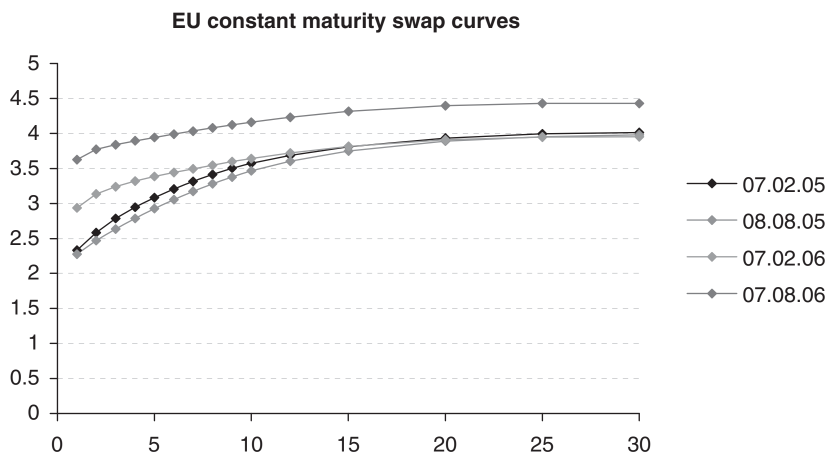

Figure 9: Illustrative swap curve movements.

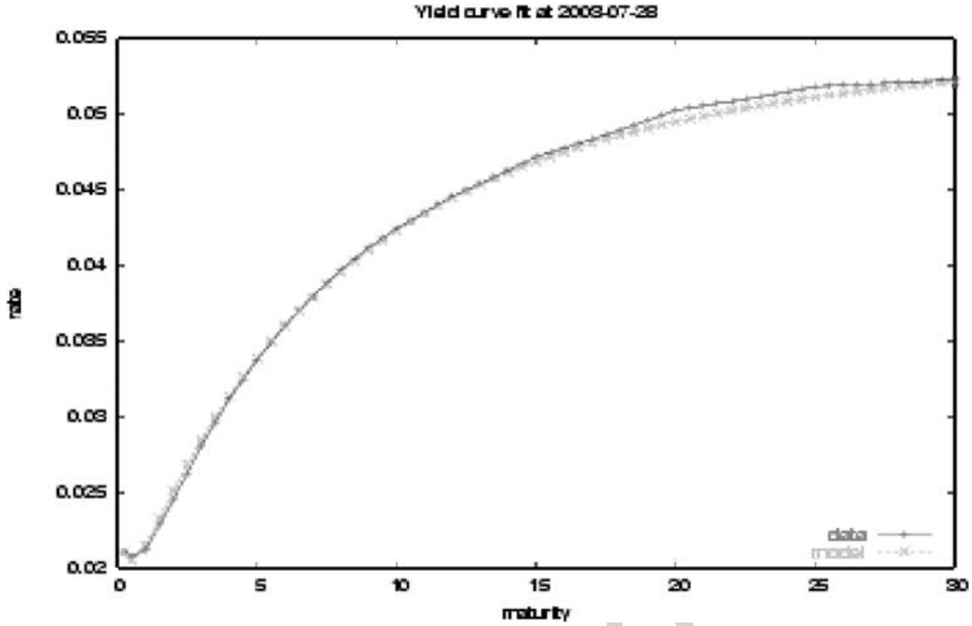

Figure 10: Yield curve fit for 28 July 2003.

between 10 and 2 year maturities for the yield curve (for zero-coupon Treasury bonds) and the swap curve are similar. However, the swap curve spread has extra volatility because of the market's changing views on general counterparty creditworthiness – essentially AA credit rating spread volatility.

Depending on the market conditions, the credit spreads of consols, similar to our example, appear to be associated with credit ratings which vary between A and BB in Standard and Poor's (S&P) terms. However, some of this market spread may be mainly because of the individual investors who bought and are currently trading these perpetual securities, evaluating them as relatively long but *finite* lived bonds. For example, currently valuing the perpetual bond as having a 25 year maturity *without* credit risk produces a discount to face value of about 30 per cent, near the current trading range.

The holder of a CMS-spread consol bond is in effect giving the issuer a levered call option on the flattening of the yield curve – that is, the *decrease* of the spread – which normally follows sharp rises in short term rates. Global macroeconomic conditions, in the late 2004 to early 2005 period, in which these contracts were issued clearly indicated sharply increasing short rates, a process that had already begun in the United States at the time and followed in the EU only shortly thereafter. Moreover, this was recently the situation because of credit market turmoil, with short rates extremely high. However, the situation from January 2011 forward is naturally unclear at this point. There exists considerable asymmetry of information between issuer and bondholder and the latter may genuinely be convinced that 'the yield curve will not become significantly flatter' in the long run to yield around 6 per cent in perpetuity.

Calibration

The data used to calibrate the model consist of freely available *daily* euro 3-and 6-month LIBOR and Euro swap data with the same maturities as in the section 'Estimation procedure' from the start of 1999 to the end of 2007, a total of 2133 observations.[9] Although the historical EU data often used by banks go back to 1992, these data have had to be constructed before the introduction of the euro in 1999. In any event, missing the sharp short rate rises of the early 1990s in our data will tend to make our valuation estimates conservative. We again interpolate the swap curve linearly to obtain swap rates at all maturities, then use the 3- and 6-month EU LIBOR rates and the swap curve to recursively back out a risk-free zero-coupon bond yield curve from the basic par swap pricing equation for each day. These derived data are the input data for estimation of the parameters of our three-factor yield curve model (which we shall do at the date of bond issue and a more recent date). From this, we can compute the yield curve based on the posterior mean for the three factors **R**, **X** and **Y** at historical dates in our data and compare this to the actual yield curve deduced from the (linearly interpolated) historical swap curve on that day. This is shown in Figure 10 for a representative date, 28 July 2003, after calibration to the data up to consol bond inception at 28 January 2005.[10]

For such calibrations on daily data, the parameter estimates for the model have similar characteristics to those given for weekly data in Tables 1 and 2 in the section 'Estimation procedure'. Although long run means and mean reversion speeds are quite stable, market prices of risk and volatilities vary with calibration end date. Moreover, as we shall see below, the resulting consol bond Monte Carlo valuations vary considerably with the initial yield, and corresponding swap, curves – actually their model approximations at the calibration end date – from which forward simulation paths for valuation on that date begin.

Ignoring for the moment the ability of the bank to call (cancel) the bond to find its fair price in the absence of credit risk, we simulate the swap curve forward under the risk adjusted probabilities (that is, with factor market prices of risk set to 0) using our interest rate model.

We compute the floating payment on each simulated scenario at each payment date and average across scenarios the total of the discounted payments along each random scenario. This is the standard Monte Carlo pricing methodology for European-style financial instruments. We used 50 000 paths for pricing the contract.

Clearly, the present value of an infinite stream of payments from this consol cannot be obtained mathematically and must be valued numerically over a finite horizon. We chose this horizon by the criterion of the maturity at which the present value at inception of the remaining coupon payments thereafter (assumed to be at the 10 per cent cap and discounted conservatively at 2.5 per cent per annum) is less than 1 per cent of face value, which occurred at 241 years.

With the right to cancel in place, the fair value is given by the expected discounted value of the sum of the coupon payments with the risk-neutral probabilities under the assumption that the bank uses an optimal call strategy. As determining the exact optimal cancellation rule is computationally difficult, we use a sub-optimal cancellation rule derived using the popular method of Andersen (1999). Owing to the fact that only the bank has the right to cancel, the sub-optimality of our cancellation strategy may lead to an *overestimation* of the value of contract from the viewpoint of the bondholder.

In brief, Andersen's method relies on a *score* function $s_t(r, x, y)$, which should be low if cancellation is likely to be correct and seeks a cancellation rule of the form: cancel if $s_t < s_t^*$. The exercise thresholds s_t^* are determined recursively based on a *separate* set of random paths for $(\mathbf{R}, \mathbf{X}, \mathbf{Y})$. We used 10 000 paths to estimate the optimal cancellation thresholds. Andersen proposed a simple method for determining good values for s_t^*. For our calculations, we take s_t^* to be the discounted value of all the remaining swap payouts to the bank under the assumption that $(\mathbf{R}, \mathbf{X}, \mathbf{Y})$ evolves according to its expected path.

We further improve the cancellation strategy as follows. Before evaluating the score function, we compute an accurate approximation to the expected value of the next payout (as this is the payment committed to by opting not to cancel the contract at this time) by linearizing the expression for the spread (CMS10−CMS2) at the end of the next period as a function of $(\mathbf{R}, \mathbf{X}, \mathbf{Y})$. This leads to an integral involving two correlated Gaussian random variables, which can be evaluated in closed form. If the expected next net return to the bank on the face value, relative to the coupon paid to the bondholder, is positive, it cannot be correct to call the bond and it is better to wait for at least one more coupon payment.

To handle the credit spread because of the creditworthiness of these subordinated consol bonds, we assume a *constant* per annum default rate appropriate to the credit class. This class could possibly be defined from market conditions (current yield curve) and similar instruments trading at different discounts because of different terms (nominal rates over 3-month euro LIBOR or EURIBOR). We might therefore have chosen to use the corresponding default rate at a constant 2.3 per cent per annum, which represents the margin over EURIBOR of the interest rate paid for similar consol bonds issued at par but on more favourable terms – and currently trading near par – by other institutions than our issuing bank in the same period. This corresponds to a S&P BB credit rating historical default rate. However, this default rate is inconsistent with the A credit rating initially assigned to this bond, and below we shall actually approximate the credit discount *implied* by the market for a 241 year maturity bond.

NPV value at risk

Value at risk (VaR) can be computed at any point in time for the bondholder from a *simulated distribution* of the *present value* (PV) of all future (*net*) payments of the deal treating the initial face value payment as a sunk cost to the bond holder. We compute VaR for the deal at inception and about 3 years later using exactly the same Monte Carlo methodology as that used for pricing, except that the market probabilities, involving estimates of the constant market prices of risk for the three factors, are used. We then compute normalized histograms of deal PVs and find the 99 per cent VaR level (relative to 1 representing par or face value) for both the issuing bank and the bondholder. As the factor market prices of risk are in fact *processes*, the standard deviations of their *constant* estimates are high relative to those of other parameters. Moreover, the estimates of the deal present value distributions are sensitive to the estimated value of the market price of risk for the short rate used to discount future payments (Cairns, 2004). We have attempted to overcome this effect in the estimation procedure by penalizing deviations from the (estimated) long run (asymptotic) short rate, which can be obtained in closed form as a function of the model parameters and is estimated from the historical 3-month EURIBOR and euro LIBOR rates.

Cash flow analysis

The swap rates for CMS2 and CMS10 (Figure 11) evolved significantly over the 2 years 2005 and 2006, which moved the spread beyond (that is, to 0.006 on 21 December 2006) its historical minimum value up to 2004 of 0.3250 per cent. This has had a dramatic effect on the forecast floating rate coupon payments to the bondholder after the 5-year 6 per cent fixed rate period (see Figure 13).

Figure 12 presents the results of simulation of the coupon payments with the mean and range given by one standard deviation, which seems beneficial for the bondholder at inception on 28 January 2005, assuming *no bond call or default* by the bank.

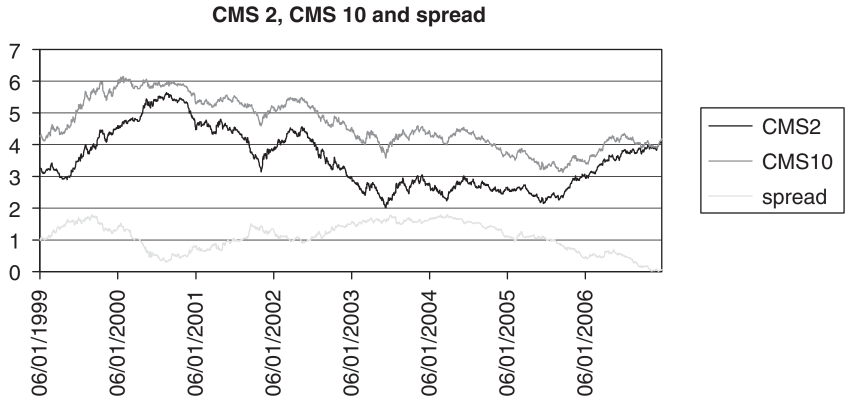

Figure 11: Base rate and spread evolution.

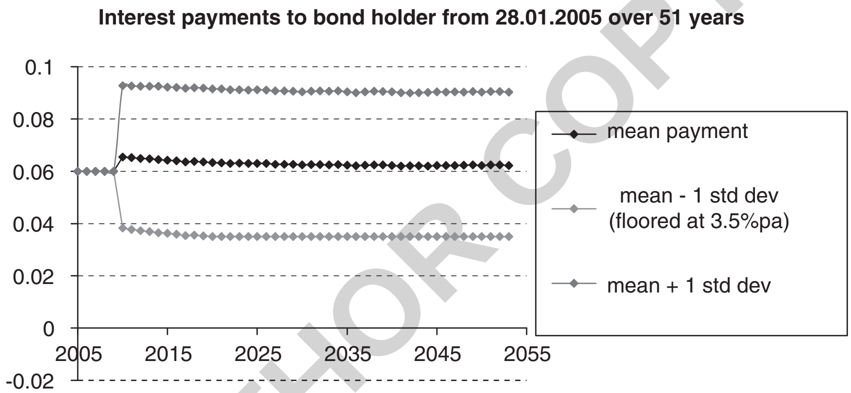

Figure 12: View of forward coupon payment distribution on 28 January 2005.

Our model predicts that the evolution of this distribution of net payments from inception is not symmetrical about its mean, which falls slightly over the life of the bond. Note that the first five points in Figure 12 represent the first five fixed annual payments to the bondholder.

Figure 13 shows the simulation of the coupon payments for the bond, assuming no call or default by the bank, from the model calibrated on data to 20 December 2007 – a year when significant changes in the spread occurred (see Figure 11). In fact, the long-run stationary distribution of the spread under the market probabilities is 1.03 per cent per annum with corresponding 3-month short rate of 3.11 per cent per annum. This spread is similar to the 1.47 per cent historical average spread from 1995 to 2005 but lower because of more recent history in which short rates were rising and long rates were depressed by global liquidity before the credit crisis. Under the risk discounted (pricing) probabilities (risk-neutral measure), the spread becomes −3.3 basis points with short rate 8.42 per cent per annum. Market moves over the 2 years from inception have been in a direction, which has made the non-credit-risk adjusted CMS-spread bond deal costly in the outcome for the bondholder at 20 December 2007 at now less than 6 per cent per annum in expectation.

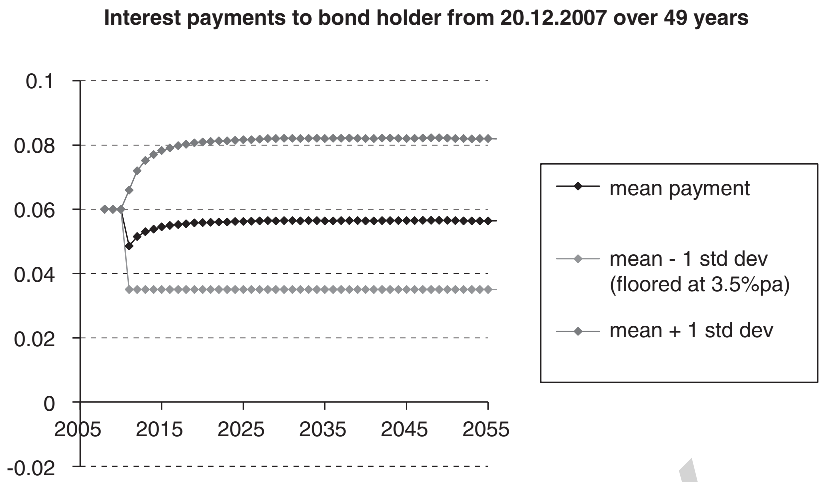

Figure 13: View of forward coupon payment distribution on 20 December 2007.

Deal valuations and values at risk

Market price of bond at inception (8 January 2005)	86.05 per cent of face value or €*1.291 million*
Six standard deviation pricing uncertainty (99.7 per cent confidence interval):	85.57–86.53 per cent of nominal or €1.284–€1.298 million
The *call option* at inception is *always used* prospectively	
Value at risk at inception to investor at the 99 per cent level:	46.27 per cent of face value recovered, that is 53.73 per cent or €*806 k lost*
Market price of bond (20 December 2007) after two 6 per cent coupon payments	80.06 per cent of face value or €*1.207 million*
Six standard deviation pricing uncertainty (99.7 per cent confidence interval):	79.67–80.44 per cent of face value or €1.195–€ 1.206 million
Value at risk on 20 December 2007 to investor at 99 per cent level:	48.97 per cent of face value recovered, that is 51.03 per cent or €*765 k lost*

A number of observations are immediate. First, the deal was not initially fairly priced at par. Approximately 14 per cent of face value of €1.5 million or €210 k was collected up front from the bond holder by the bank. The initial fair price of 86 per cent of face value is a higher than current market prices for these bonds, which are currently in the range of 60–65 per cent – perhaps because of higher credit risk or the individual investor finite horizon effect discussed above.

Second, in the absence of default, the bank's call (cancellation) option is optimally *always used*. The call date varies by scenario from 5 to 224 years from inception with an average of about 18 years.

Third, the 99 per cent VaR to the bondholder involves considerable losses and depends critically on market conditions. Note, however, that 30 per cent of the face value is recovered in the five fixed payments before discounting. The situation is illustrated graphically in Figures 14 and 15, which give the distributions of PV of future payments as a proportion of nominal (1 represents face value). These figures show the asymmetry in these distributions with a long, thin upside tail in favour of the bondholder and a significantly *probable downside* in favour of bank. From inception at 28 January 2005 to 20 December 2007, the expected PV has been reduced significantly from 1.154 (115.4 per cent of face value) to

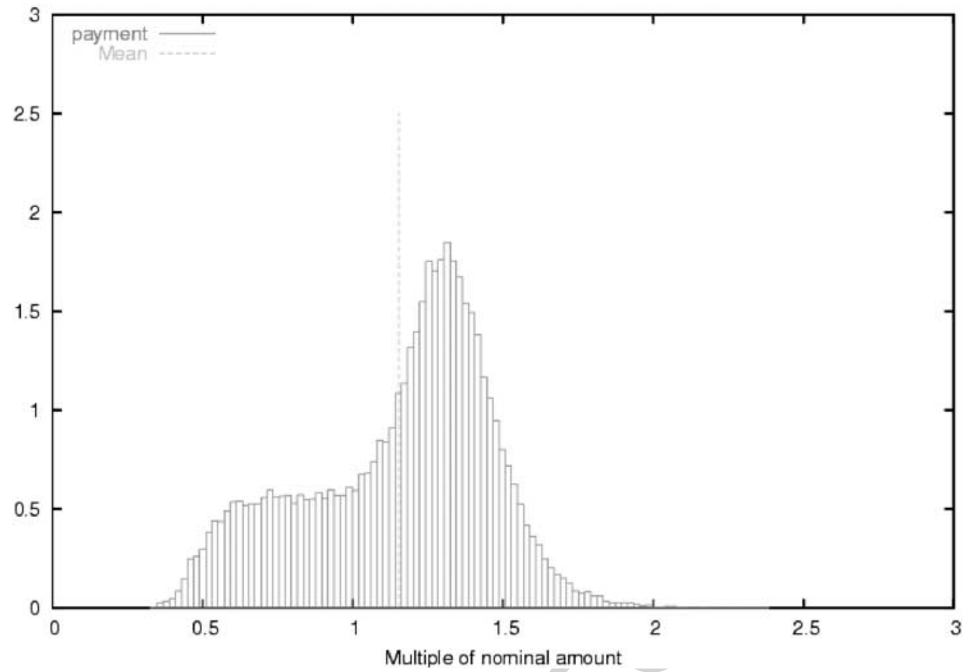

Figure 14: Distribution of total discounted payments to investor in multiple of face value at 28 January 2005.

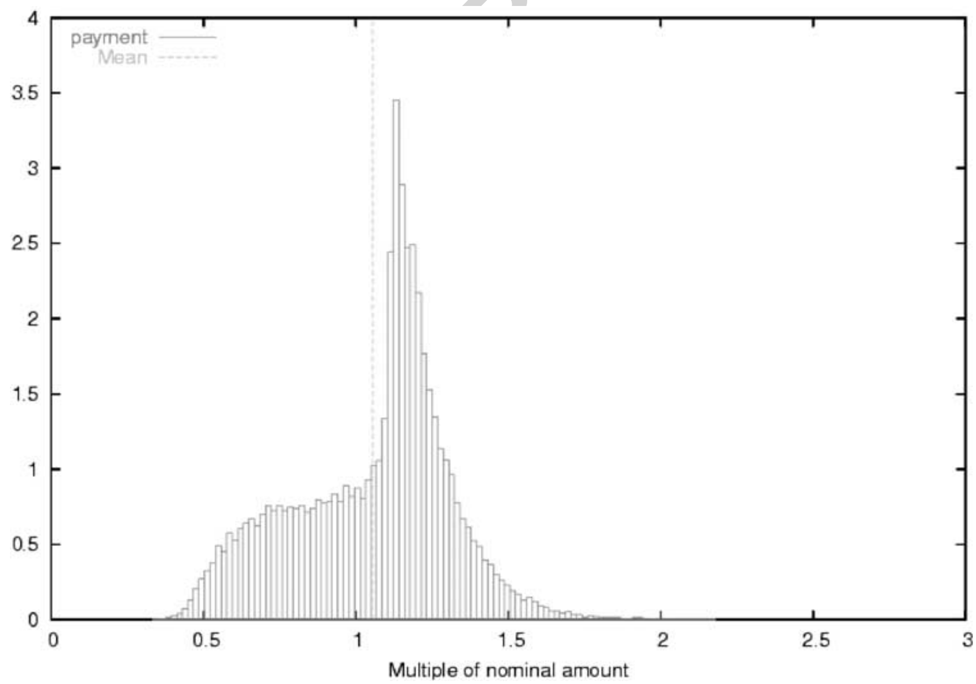

Figure 15: Distribution of total discounted payments to investor at 20 December 2007.

1.055 (105.5 per cent), although the more appropriate median present value has been reduced somewhat less.

Comparison with risk-free bonds

By comparison with the 115.4 per cent PV of future payments at inception of the actual contract on 28 January 2005 (Figure 14), the corresponding figure for an 18-year maturity 6 per cent fixed coupon risk-free bond (with no call option) is 120.6 per cent. The 5.2 per cent reduction for the consol bond represents the balance between the average effects of the call option and the potential for coupon payments near the floor (reducing) and the potential for higher coupon payments of up to 10 per cent over periods when the bank optimally calls the perpetual bond later than the average 18 years (enhancing). Nearly 3 years later, on 20 December 2007 (Figure 15) when the PV of future payments is only 105.5 per cent, the corresponding reduction relative to a 15-year maturity 6 per cent risk-free bond exceeds 15 per cent. At this date, the likelihood of the bondholder not even recovering the original investment has risen to nearly 50 per cent.

Credit risk analysis

Finally, let us consider the effects of credit risk on the current market price of this consol bond. Taking account of credit risk for such a perpetual bond from a necessarily (short) finite amount of historical default data is fraught with error. We have chosen to use the maturity of 241 years of our finite maturity approximation to obtain at least a plausible value for the credit risk discount. The difference between our market risk valuation of 80 per cent of face value on 20 December 2007 and the market value of the bond on that day of 60 per cent allows 20 per cent of face value as credit risk discount and/or behavioural finite horizon and tax effects. Assuming that this figure is essentially because of *pure credit risk* implies that the market on that day was assuming approximately a default discount rate of 8.3 bp per annum,[11] which would lead to a credit risk discount over 241 years of 20.1 per cent.

The only rational explanation for the purchase of this representative consol is that bondholders believed that their coupon payments would genuinely not be significantly reduced from the first five at 6 per cent. As we have seen (in Figure 12), this would have been a reasonable expectation based on history at inception, but actual market outcomes – possibly revealed to the issuing bank in the forward economic and market views at inception – have further moved strongly against the bondholders (see Figure 13). The result is that an investor is left holding an illiquid credit risky perpetual bond, which is currently trading at a 30 per cent to 40 per cent loss on their investment with the prospect in 2 years time of possibly receiving only 3.5 per cent annual coupons because of CMS-spreads of only a few basis points.

Obviously, these credit risky structural floating rate consol bonds provide investors with returns far inferior to risk-free fixed rate bonds of comparable expected maturities issued in the same period. However, the complexity of these instruments, which require sophisticated Monte Carlo analysis to price, has by and large been ignored by investors. Indeed, initially investors appear to have treated these securities naively, and sub-optimally, as short maturity risk-free fixed rate bonds that would be called by banks soon after all their initial fixed rate payments were made.

CONCLUSION

The objective of this article is to specify a model that captures the salient features of the whole term structure, rather than one that just focuses on the short-term interest rate. It also has to be tractable in order to form a basis for asset pricing applications and forward simulations for asset liability management. To this end, we consider a Gaussian three-factor continuous-time model within the affine class with a closed-form solution for bond prices.

For our empirical analysis, the model is expressed in a state-space formulation, which allows us to take into account both the cross-sectional and time-series information contained in the term structure data and to use the Kalman filter and numerical likelihood maximization recursively to estimate the parameters.

The model explains the cross-section of interest rates well with reasonably small yield errors. We also show that in forward simulations this model gives rise to a wide and realistic range of future interest rate scenarios, as shown by both backtest and simulation results, involving flattening/steepening/inversion of the yield curve.

We apply the model to pricing perpetual callable consol bonds with structured floating coupon payments, based on the 10–2 year CMS-spread, using forward simulations over a 241-year horizon with a daily time step. As a result, we find these credit risky floating rate consol bonds issued by banks to raise Tier 1 capital in the 2004–2005 period to be initially mispriced and with lower expected yields than comparable finite horizon sovereign fixed coupon bonds.

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NOTES

1. We use boldface throughout the paper to denote random or conditionally random entities.
2. This assumption implies that the conditional variance of yield changes is constant over time. A number of studies concerned with the relatively short term have found that yield changes are conditionally heteroscedastic, cf. Ball and Torous (1996). Fong and Vasicek (1991) introduced stochastic volatility to represent this situation, whose relevance to the long run is questionable, for pricing (see also Litterman *et al*, 1991 and Andersen *et al*, 2004).
3. Alternatively, De Jong (2000) presents a general way to obtain the exact discrete-time state distributions in affine class models. As the benefits are unclear for our purposes and simulation complexity increases, we have not pursued this approach here.
4. A *par interest rate swap* is a standard contract between two counterparties to exchange cash flows. At set time intervals termed *reset dates,* one pays a predetermined *fixed* rate of interest on the *nominal* value, the other a *floating* rate, until the *maturity* date of the contract. The floating leg of swap fixes the interest rates for each payment at the rate of a published interest rate. The fixed rate, known as the *swap rate,* is that interest rate, which makes the fair value of the par swap 0 at inception. Thus, the cash flows of the two legs of a par swap are those of a pair of bonds with face value the swap nominal, one fixed rate and the other floating rate.
5. But, see the section 'Pricing consol bonds' in which more recent data up to 2008 are used.
6. We also evaluated the quadratic interpolation but deemed the negligible improvement in accuracy not worth the considerable increase in computational burden.
7. Given the relatively low yield volatilities depicted in Figure 6 and the yield levels in Figure 5 we concluded that the probability of negative yields with our Gaussian model under the market measure is negligible. Using values from the data of Figure 5 and 6, these correspond to a minus 10 standard deviation event. Our decision is borne out by the representative paths in Figures 7 and 8 and, in fact, none of the 500 scenarios simulated produced negative yields over the out-of-sample period. However, see Abu-Mostafa (2001) for a technique for reducing this probability over longer simulation horizons.
8. For example, in 2005 Deutsche Bank issued a €900 million tranche of bonds at par to *face value* or *nominal*. This revives an instrument that has not been in favour since the Russian Revolution, when Tsar Nicholas' consols became worthless, although UK consols initiated in the eighteenth century are still in existence (with reduced fixed coupon). There is little current literature on their pricing when coupon rates are floating.
9. We use daily data for consol bond valuation to conform to market practice by issuers who value fixed income instruments incorporating yield curve data on (or just before) the day of sale. Our example here is representative of a number of consol bonds we have valued initially on different dates in the period 2004–2006.
10. Note that these fits on representative days do not always accurately capture the long end of the yield curve, which might require a fourth factor. They are, however, acceptably accurate up to 10-year maturity and in any event generally err on the conservative side, by producing lower discount rates.
11. This corresponds to the historical *4-year* S&P cumulative default rate for the bond's A rating which suggests that the market was optimistic regarding the bank's possible default on the contract over possibly nearly two and a half centuries.

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